APPLICATION OF THE MONTE CARLO METHOD TO THE PROBLEM OF FLOW IN THE BOUNDARY LAYER OF A TWO-DIMENSIONAL TURBULENT JET

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A precise solution is found to the problem of flow in the boundary layer of a two-dimensional turbulent jet. The method of [1] is used here.

The flow of an incompressible fluid in the boundary layer of a two-dimensional turbulent jet is described by the following equations and boundary conditions:

$$u_\beta u_{\alpha,\beta} + P_{,\alpha} = \tau_{\alpha\beta,\beta}; \tag{1}$$

$$u_{\sigma,\sigma} = 0; \tag{2}$$

$$\tau_{\alpha\beta} = -l_{\alpha\beta}^2 u_{1,2}^2; \tag{3}$$

$$\left.\begin{aligned} &u_1|_{\Gamma_0} = 1, \\ &u_2|_{\Gamma_0} = u_{1,2}|_{\Gamma_0} = u_1|_{\Gamma_1} = u_{1,2}|_{\Gamma_1} = 0 \end{aligned}\right\}. \tag{4}$$

The summation role has been adopted here with respect to the repetitive Greek letter subscripts which assume the values 1, 2. The symbol after a comma denotes the derivative with respect to the given coordinate and u_α are the velocity components. On the basis of the similarity theory, we assume that u_α, P, and $x_1^{-1} l_{\alpha\beta}$ are functions of the ratio $\eta = x_1^{-1}x_2$. The flow function will be sought in the form

$$\psi = x_1 F(\eta).$$

Adopting the obvious rules of differentiation

$$A_{,1} = -A_{,\eta}\, \eta x_1^{-1};\ A_{,2} = A_{,\eta}\, x_1^{-1}$$

and considering that

$$u_1 = \psi_{,2},\ u_2 = -\psi_{,1},$$

$$l_{\alpha\beta} = \mu_{\alpha\beta}(\eta)\, x_1,$$

one can reduce systems (1)-(4) to a single third-order ordinary differential equation (a prime sign indicates a derivative with respect to η)

$$\begin{aligned} Q = \left[(1-\eta^2)\ \mu_{12}^2 + \eta\left(\mu_{22}^2 - \mu_{11}^2\right)\right] F''' + \left[(1-\eta^2)\, \mu_{12}\, \mu_{12}' \right.\\ \left. + \eta\left(\mu_{22}\, \mu_{22}' - \mu_{11}\, \mu_{11}'\right)\right] F'' - \frac{1}{2}(1+\eta^2) F = 0. \end{aligned} \tag{5}$$

The boundary conditions (4) become

$$F|_{\alpha=0} = \eta_0;\ F'|_{\alpha=0} = 1;\ F''|_{\alpha=0} = F'|_{\alpha=1} = F''|_{\alpha=1} = 0. \tag{6}$$

Here $\alpha = (\eta-\eta_0)/(\eta_1-\eta_0)$; $\alpha \in [0; +1]$, while parameters η_0 and η_1 define the location of boundaries Γ_0 and Γ_1. To conditions (6) we add two others:

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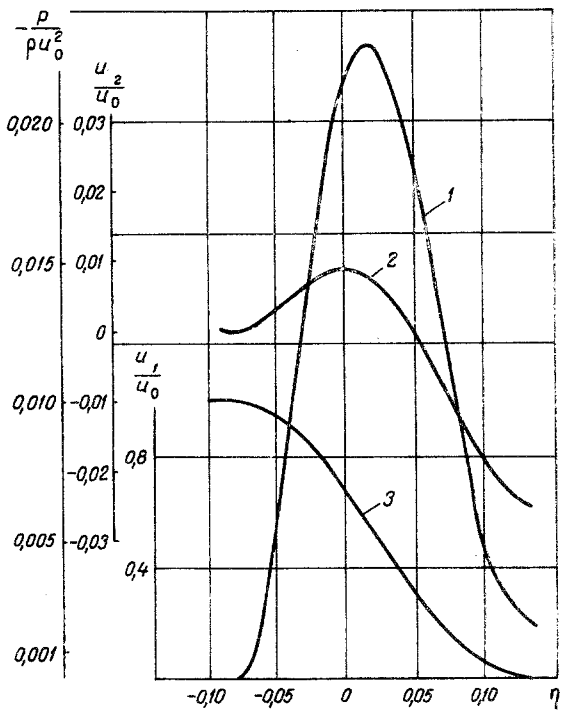

Fig. 1. Functions of $\eta = x_2/x_1$: 1) $-P/\rho u_0^2$; 2) u_2/u_0; 3) u_1/u_0 (given: $\mu_{11} = 0.030$, $\mu_{12} = \mu_{21} = 0.014$, $\mu_{22} = 0.020$, $\nu = 50$; obtained: $\eta_0 = -0.0776, \eta_1 = +0.1382, k_0 = -130.6$, $k_1 = +66.51$).

$$F'''|_{\alpha=0} = k_0;\ F'''|_{\alpha=1} = k_1,$$

where k_0, k_1 are constants not yet known.

The solution to the boundary-value problems (5)-(6) will be sought in the form of a polynomial

$$F = \varphi_{-1} + \varphi_0\alpha + \frac{1}{2}\varphi_1\alpha^2 + \frac{1}{3}\varphi_2\alpha^3 + \frac{1}{4}\varphi_3\alpha^4 + \frac{1}{5}\varphi_4\alpha^5 + \frac{1}{6}\varphi_5\alpha^6, \tag{7}$$

whose coefficients are defined as follows:

$$\varphi_{-1} = \eta_0,\ \varphi_0 = \Delta = \eta_1 - \eta_0,\ \varphi_1 = 0,\ \varphi_2 = \frac{1}{2}\Delta^3 k_0,$$

$$\varphi_3 = \left[-10 + (k_1 - 3k_0)\frac{\Delta^2}{2}\right]\Delta,$$

$$\varphi_4 = \left[15 + (3k_0 - 2k_1)\frac{\Delta^2}{2}\right]\Delta,$$

$$\varphi_5 = \left[-6 + (k_1 - k_0)\frac{\Delta^2}{2}\right]\Delta.$$

In this case the boundary conditions are satisfied automatically and the four parameters η_0, η_1, k_0, k_1 remain unconstrained.

In accordance with [1], the values of these parameters at which the quantity

$$\Phi = N^{-1}\sum_{s=1}^{N} Q_s^2 \tag{8}$$

becomes minimum will be considered optimal. Here N is the total number of control points on the interval $[\eta_0, \eta_1]$. Their coordinates were calculated by the formula

$$\eta^{(s)} = \eta_0 + sN^{-1}(\eta_1 - \eta_0),$$

with s denoting the consecutive number of a point and Q_s denoting the magnitude of the error incurred by inserting polynomial (7) into Eq. (5) at $\eta = \eta^{(s)}$. In accordance with [3], the following values were taken for $\mu_{\alpha\beta}$: $\mu_{11} = 0.03$, $\mu_{12} = 0.014$, $\mu_{22} = 0.02$. Equation (8) was minimized by the method of random tracking [2]. On the basis of optimal values for η_0, η_1, k_0, and k_1 with the aid of expressions

$$u_1/u_0 = F'; \; u_2/u_0 = \eta F' - F;$$

$$\frac{P}{\rho u_0^2} = 2 \int\limits_{\eta_0}^{\eta} \left[\frac{1}{2} \eta F + \eta \mu_{12} \left(\mu_{12}' F'' + \mu_{12} F''' \right) - \mu_{22} \left(\mu_{22}' F'' + \mu_{22} F''' \right) \right] F'' \, d\eta$$

curves have been plotted as shown in Fig. 1. They come close to the empirical curves in [3].

Evidently, a further refinement is possible by increasing the number of terms in expression (7) and by changing conditions (6) to more stringent ones (with transverse flow also taken into account).

NOTATION

x_1, x_2	are the Cartesian coordinates;
u_1, u_2	are the velocity components;
P	is the pressure, referred to a plane surface;
$\eta = x_2/x_1$	is the dimensionless coordinate;
Γ_0, Γ_1	are the inside and outside boundary of a layer;
η_0, η_1	are their respective coordinates;
$\alpha = (\eta - \eta_0)/(\eta_1 - \eta_0)$	is the coordinate referred to interval [0, +1];
$\psi = x_1 F(\eta)$	is the flow function;
$\tau_{\alpha\beta} = -l_{\alpha\beta}^2 u_{1,2}^2 = -\mu_{\alpha\beta}^2(\eta) x_1^2 u_{1,2}^2$	are the components of the tensor of turbulent stresses, referred to the density;
F	is the power polynomial representing the solution;
$\varphi_{-1}, \varphi_0, \varphi_1, \ldots, \varphi_5$	are the coefficients of the power polynomial;
Φ	is the functional to be minimized.

LITERATURE CITED

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